Example 1: (D_n) In D_n, we have the subgroup generated by a rotation r through the angle $\frac{2\pi}{n}$. Let $H = \langle r \rangle$. Then

$|H| = n$, $|D_n| = 2n$,

so $[D_n : H] = 2$.

By your homework,

$H \trianglelefteq D_n$.

Then D_n/H is a group.

Which group is it?

There are precisely 2 cosets, so

D_n/H is isomorphic to $\mathbb{Z}_2$.

The quotient map φ can be regarded as a map from D_n to $\mathbb{Z}_2$, where

$\varphi(r^k) = [0] \quad \forall\, k \in \mathbb{N}$

$\varphi(\text{any flip}) = [1]$.

Homomorphism Theorem : (a.k.a. the First Isomorphism Theorem)

Let $\varphi: G \to H$ be a surjective homomorphism. Let $N = \ker(\varphi)$.

Then G/N is isomorphic to H.

(most important of the theorems)

Correspondence Theorem: (a.k.a. the Second Isomorphism Theorem) Let $\varphi: G \to H$ be a surjective homomorphism and let $N = \ker(\varphi)$

Then

1) The map $K \mapsto \varphi^{-1}(K)$ provides a bijective correspondence between subgroups K of H and subgroups of G containing N.

2) Under this map, normal subgroups of H correspond to normal subgroups of G.

Factorization Theorem: (a.k.a. the Third Isomorphism Theorem) Let $\varphi: G \to H$ be a surjective homomorphism. Let $\ker(\varphi) = N$. Suppose $K \leq N$ and $K \trianglelefteq G$. If $\pi: G \to G/K$ is the quotient map, then $\exists$ a surjective homomorphism $\bar{\varphi}: G/K \to H$ such that

$$\bar{\varphi} \circ \pi = \varphi.$$

(φ factors as $\bar{\varphi} \circ \pi$)

Diamond Isomorphism Theorem: (a.k.a. the Fourth Isomorphism Theorem)

Let $\varphi: G \to H$ be a surjective homomorphism. Let $N = \ker(\varphi)$.

If $K \leq G$, then

1) $\varphi^{-1}(\varphi(K)) = KN$

$$:= \{xy \mid x \in K, y \in N\}$$

2) $KN \leq G$

3) KN/N is isomorphic to $\varphi(K)$, which is isomorphic to $K/K \cap N$

Picture

$$\begin{tikzcd}[column sep=small]
 & KN \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[dr, dash] & & N \arrow[dl, dash] \\
 & K \cap N &
\end{tikzcd}$$